SHRI VISHWAKARMA SKIL UNIVERSITY

(CLASS NOTES)

SUBJECT: ZDSC-103(APPLIED MATHEMATICS-I)

FACULTY NAME: Dr. M.K. SRIVASTAV

UNIT-3: ALGEBRA-I

Topic 3.1: Partial Fraction

An algebraic fraction is a fraction in which the numerator and denominator are both polynomial expressions.

Example:

1. $\frac{3x}{x^2+2x+1}$
2. $\frac{1-x^3}{x^4-2x^2+1}$

} Proper fraction (The numerator is a polynomial of lower degree than the denominator)

3. $\frac{x^4+3x^3+x}{x^2-x+7}$
4. $\frac{x+1}{1-x}$

} Improper fraction (The numerator is a polynomial of higher degree than the denominator)

Expressing a fraction as the sum of its partial fractions

To express a single algebraic fraction into the sum of two or more single algebraic fraction is called **Partial fraction resolution** and these algebraic fraction are called **Partial fractions.**
The method for computing partial fraction decompositions applies to all partial functions with one qualification:

- **The degree of the numerator must be less than the degree of the denominator (i.e. for Proper fraction only)**

Rule **1.** For a linear term $ax+b$ in the denominator, we get a partial fraction as: $\frac{A}{ax+b}$

2. For a repeated linear term in the denominator, such as $(ax+b)^3$, we get partial fractions as: $\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$

3. For a quadratic term ax^2+bx+c in the denominator, we get we partial fractions as: $\frac{Ax+B}{ax^2+bx+c}$

4. For a repeated quadratic term in the denominator, such as $(ax^2+bx+c)^2$, we get partial fractions as: $\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

Example: I. $\frac{x-1}{(x+2)(x-7)}=\frac{A}{(x+2)}+\frac{B}{(x-7)}$

II. $\frac{2x^2+1}{x^3-x^2-8x+12}=\frac{2x^2+1}{(x-2)^2(x+3)}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x+3)^2}$

III. $\frac{x^2+1}{(x^2+x+2)(x+7)}=\frac{Ax+B}{(x^2+x+2)}+\frac{C}{(x+7)}$

IV. $\frac{x^2+1}{(x-1)(x+2)(x^2+2x+5)^2}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{Cx+D}{(x^2+2x+5)}+\frac{Ex+F}{(x^2+2x+5)^2}$

Computing the coefficients:

First we determine the right form for the partial fraction decomposition of an algebraic fraction, then to compute the unknown coefficients *A*, *B*, *C*,... we use two methods for this purpose. We will now look at both methods for the decomposition of:

$$\frac{2x-1}{(x+2)^2(x-3)}$$

Applying above rules:

$$\frac{2x-1}{(x+2)^2(x-3)}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{(x-3)}$$

$$\Rightarrow \frac{2x-1}{(x+2)^2(x-3)}=\frac{A(x+2)(x-3)+B(x-3)+C(x+2)^2}{(x+2)^2(x-3)}$$

$$\Rightarrow 2x-1=A(x+2)(x-3)+B(x-3)+C(x+2)^2 \qquad ......(1)$$

- In the ***first method***, we substitute different values for x in to the equation no. (1) and deduce the values of A, B, C.

So putting $x=3$ in equation (1) we have:

$$2\times3-1=A(3+2)(3-3)+B(3-3)+C(3+2)^2$$

$$\Rightarrow 5=25C$$

$$\Rightarrow C=\frac{1}{5}$$

Putting $x=-2$ in equation (1) we have:

$$2\times(-2)-1=A(-2+2)(-2-3)+B(-2-3)+C(-2+2)^2$$

$$\Rightarrow -5=-5B$$

$$\Rightarrow B=1$$

Putting $x=0$ in equation (1) we have:

$$2\times0-1=A(0+2)(0-3)+B(0-3)+C(0+2)^2$$

$$\Rightarrow -1=-6A-3B+4C$$

$$\Rightarrow -1=-6A-3(1)+4\left(\frac{1}{5}\right)\Rightarrow A=-\frac{1}{5}$$

Therefore: $\frac{2x-1}{(x+2)^2(x-3)} = \frac{-1/5}{(x+2)} + \frac{1}{(x+2)^2} + \frac{1/5}{(x-3)}$

In the ***second method***, we expand the right side and collect like terms:

$$2x-1 = Ax^2 - Ax - 6A + Bx - 3B + Cx^2 + 4Cx + 4C$$
$$= (A+C)x^2 + (-A+B+4C)x + (-6A-3B+4C)$$

For these polynomials to be equal, their coefficients must be equal, leading us to the system of equation:

$0 = A + C$ from the x^2 term

$2 = -A + B + 4C$ from the x term

$-1 = -6A - 3B + 4C$ from the constant term

Solving these equation, we get, $A = -\frac{1}{5},\ B = 1,\ C = \frac{1}{5}$

And therefore: $\frac{2x-1}{(x+2)^2(x-3)} = \frac{-1/5}{(x+2)} + \frac{1}{(x+2)^2} + \frac{1/5}{(x-3)}$

Topic 3.2: Permutation

Fundamental Principle of Counting

If an event can occur in *m* different ways, following which another event can occur in *n* different ways, then the total number of occurrence of the events in the given order is $m \times n$.

Example: Find the number of 5 letter words, with or without meaning, which can be formed out of the letters of the word **KUMAR**, where the repetition of the letters is not allowed.

Solution: There are as many words as there are ways of filling in 5 vacant places □□□□□ by the 5 letters, keeping in mind that the repetition is not allowed.

- The first place can be filled in 5 different ways by anyone of the 5 letters K, U, M, A, R.
- Following which, the second place can be filled in by anyone of the remaining 4 letters in 4 different ways
- Following which the third place can be filled in 3 different ways
- Following which, the fourth place can be filled in 2 ways.
- Following which, the fourth place can be filled in 1 way

Thus, the number of ways in which the 5 places can be filled, by the multiplication principle, is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Note: If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 5 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $5 \times 5 \times 5 \times 5 = 3125$.

Example: Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution: There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = 4 × 3 = 12.

Example: How many 2 digits odd numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution: There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 1, 3 and 5 only and this can be done in 3 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 3 × 5 = 15.

Factorial of Number

The factorial of a natural number *n* is denoted by $n!$ and it is the product of first *n* natural numbers.

Therefore, $n! = 1\times 2\times 3\times .....\times (n-1)\times n$

Or $n! = n\times (n-1)\times .....\times 3\times 2\times 1$

Example:

$0! = 1$

$1! = 1$

$2! = 2\times 1 = 2$

$3! = 3\times 2\times 1 = 6$

$4! = 4\times 3\times 2\times 1 = 24$

Clearly, $n! = n\times (n-1)!$

Example: Calculate the value of $\frac{8!}{5!}$

Solution: $\frac{8!}{5!} = \frac{8\times 7\times 6\times 5!}{5!} = 336$

Example: If $\frac{1}{5!}+\frac{1}{6!} = \frac{n}{7!}$, find *n*.

Solution: We have,

$\frac{1}{5!}+\frac{1}{6!} = \frac{n}{7!} \quad \Rightarrow \frac{1}{5!}+\frac{1}{6\times 5!} = \frac{n}{7\times 6\times 5!} \quad \Rightarrow 1+\frac{1}{6} = \frac{n}{7\times 6} \quad \Rightarrow \frac{7}{6} = \frac{n}{7\times 6}$

$\Rightarrow n = 49$

Permutation

A permutation is an arrangement in a definite order of a number of objects taken some or all at a time. Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged.

Case1. Permutations when all the objects are distinct:

The number of permutations of *n* different objects taken *r* at a time, where $0 < r \le n$ and the objects do not repeat is *n* (*n* – 1) (*n* – 2) . . . (*n* – *r* + 1), which is denoted by ^nP_r .

$$^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n$$

Example: How many 5-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution: Here order matters for example 12345 and 13245 are two different numbers. Therefore, there will be as many 5 digit numbers as there are permutations of 9 different digits taken 5 at a time. Therefore, the required 5 digit numbers = $^9P_4 = \frac{9!}{(9-5)!} = \frac{9!}{4!} = 9\times 8\times 7\times 6\times 5 = 15120$

Case2: The number permutations of *n* different objects taken *r* at a time, where repetition is allowed, is n^r.

Case3. Permutations when all the objects are not distinct:

The number of permutations of *n* objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of different kind is $\frac{n!}{p_1!\,p_2!....p_k!}$

Example: Find the number of permutations of the letters of the word ALLAHABAD.

Solution: Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements = $\frac{9!}{4!2!} = \frac{9\times 8\times 7\times 6\times 5\times 4!}{4!2!} = 7560$

Example: How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

Solution: Every number between 100 and 1000 is a 3-digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be $^6P_3$. But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from $^6P_3$ to get the required number. To get the number of such numbers, we fix 0at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is $^5P_2$. So

The required number $^6P_3 - {}^5P_2 = \frac{6!}{3!} - \frac{5!}{3!} = 6\times5\times4 - 5\times4 = 100$

Example: Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that
(i) all vowels occur together (ii) all vowels do not occur together.

Solution: (i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time.

This number would be $^6P_6 = \frac{6!}{(6-6)!} = 6!$. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time.
Hence, by the required number of permutations = 6! × 3! = 4320.

(ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangements of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together.
Therefore, the required number = 8! – 6! × 3! = 6! (7×8 – 6)
= 2 × 6! (28 – 3)
= 50 × 6! = 50 × 720 = 36000

Example: Find the value of n such that $^nP_5 = 42\,{}^nP_3,\ n > 4$

Solution: Given that $^nP_5 = 42\,{}^nP_3$

$$\Rightarrow \frac{n!}{(n-5)!} = 42\frac{n!}{(n-3)!}$$

$$\Rightarrow n\,(n-1)\,(n-2)\,(n-3)\,(n-4) = 42\,n(n-1)\,(n-2)$$

Since $n > 4$ then $n\,(n-1)\,(n-2) \neq 0$

Therefore, by dividing both sides by $n(n-1)\,(n-2)$, we get

$n^2 - 7n - 30 = 0$

$n^2 - 10n + 3n - 30 = 0$

or $(n-10)(n+3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$

<u>Topic 3.3: Combinations</u>: A collection of things, in which the order does not matter.

Example: Suppose we have a set of three letters: A, B, and C. Then how many ways we can select 2 letters from this set. Each possible selection would be an example of a combination. The complete list of possible selections would be: AB, BC, CA = 03.

Note:

$$(i)\, {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}, 0 \le r \le n$$

$$(ii)\, {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

Example: If ${}^nC_5 = {}^nC_6$, find the value of *n*.

Solution: we have

$${}^nC_5 = {}^nC_6$$

$$\Rightarrow \frac{n!}{5!(n-5)!} = \frac{n!}{6!(n-6)!} \quad \Rightarrow \frac{1}{5!(n-5)(n-6)!} = \frac{1}{6 \times 5!(n-6)!} \quad \Rightarrow \frac{1}{(n-5)} = \frac{1}{6}$$

$$\Rightarrow n = 11$$

Example: A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution: Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time.

Hence, the required number of ways = ${}^5C_3 = \frac{5!}{3!2!} = 10$

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways.

Therefore, the required number of committees = ${}^2C_1 \times {}^3C_2 = \frac{2!}{1!1!} \times \frac{3!}{2!1!} = 6$

Example: What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these
(i) four cards are of the same suit,
(ii) four cards belong to four different suits,
(iii) are face cards,
(iv) two are red cards and two are black cards,
(v) cards are of the same colour?

Solution: There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time.
Therefore, The required number of ways

$$= {}^{52}C_4 = \frac{52!}{4!48!} = \frac{52 \times 51 \times 50 \times 49 \times 48!}{4 \times 3 \times 2 \times 48!} = 270775$$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are

$^{13}C_4$ ways of choosing 4 clubs, $^{13}C_4$ ways of choosing 4 spades and $^{13}C_4$ ways of choosing 4 hearts.

Therefore, The required number of ways = $^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$

$$= 4 \times {}^{13}C_4 = 4 \times \frac{13!}{4!9!} = 2860$$

(ii) There are13 cards in each suit. Therefore, there are $_{13}C_1$ ways of choosing 1 card from 13 cards of diamond, $^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, $^{13}C_1$ ways of choosing 1card from 13 cards of clubs, $^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$$
$$= 13 \times 13 \times 13 \times 13$$
$$= 13^4$$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in $^{12}C_4$ ways. Therefore, the required number of ways $= \frac{12!}{4!8!} = 495$

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways

$$= {}^{26}C_2 \times {}^{26}C_2$$
$$= \left(\frac{26!}{2!24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards on $^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in $^{26}C_4$ ways. Therefore, the required number of ways

$$= {}^{26}C_4 + {}^{26}C_4$$
$$= 2 \times {}^{26}C_4$$
$$= 2 \times \frac{26!}{4!22!} = 29900$$

Topic 3.4: Binomial theorem for any positive integer *n*:

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

Note:

1. The above expression can also be written as: $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$
2. The coefficients ^nC_r occurring in the binomial theorem are known as binomial coefficients.
3. There are (n +1) terms in the expansion of $(x+y)^n$, *i.e.*, one more than the index (n).
4. Similarly $(x-y)^n = {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + (-1)^n\,{}^nC_ny^n$

Example: $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

For $x = 1,$ then $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + ... + (-1)^n\, {}^nC_n x^n$

For $x = 1$, then $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 + ... + (-1)^n\, {}^nC_n$

Example: Expand $(x^2 + \frac{3}{x})^4, x \neq 0$

Solution: Formula $(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + ... + {}^nC_{n-1} x y^{n-1} + {}^nC_n y^n$

Therefore:

$$(x^2 + \frac{3}{x})^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3 \left(\frac{3}{x}\right) + {}^4C_2 (x^2)^2 \left(\frac{3}{x}\right)^2 + {}^4C_3 (x^2)\left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$$

$$= x^8 + 4x^6.\left(\frac{3}{x}\right) + 6x^4.\left(\frac{9}{x^2}\right) + 4x^2.\left(\frac{27}{x^3}\right) + \left(\frac{81}{x^4}\right)$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}$$

Example: Compute $(98)^5$

Solution: Write 98 = 100-2

Therefore

$$(98)^5 = (100-2)^5$$

$$= {}^5C_0 (100)^5 - {}^5C_1 (100)^4.(2) + {}^5C_2 (100)^3.(2)^2 - {}^5C_3 (100)^2.(2)^3 + {}^5C_4 (100).(2)^4 - {}^5C_5 (2)^5$$

$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 + 5 \times 100 \times 16 - 32$$

$$= 10040008000 - 1000800032 = 9039207968$$

General Term: General term in the expansion of $(x+y)^n$ is given by

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

Middle Terms: **(1)** If n is even, then the middle term is $\left(\frac{n}{2}+1\right)^{\text{th}}$ term.

(2) If n is odd, then the middle terms are $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ term.

Example: In the expansion of $\left(x + \frac{1}{x}\right)^{2n}$, where x ≠ 0,

The middle term is $\left(\frac{2n+1+1}{2}\right)^{\text{th}} = (n+1)^{th}$ term. ($2n$ is even)

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant)

Example: Find a if the 17th and 18th term of the expansion $(2+a)^{50}$ are equal.

Solution: The $(r+1)^{th}$ term of the expansion $(x+y)^n$ is

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

For the 17th term, we have $r + 1 = 17$

$\Rightarrow r = 16$

therefore, $T_{17} = T_{16+1} = {}^{50}C_{16}(2)^{50-16}(a)^{16} = {}^{50}C_{16}(2)^{34}(a)^{16}$

similarly, $T_{18} = T_{17+1} = {}^{50}C_{17}(2)^{50-17}(a)^{17} = {}^{50}C_{17}(2)^{33}(a)^{17}$

now $T_{17} = T_{18}$

so ${}^{50}C_{16}(2)^{34}(a)^{16} = {}^{50}C_{17}(2)^{33}(a)^{17}$

Therefore $\frac{{}^{50}C_{16}(2)^{34}(a)^{16}}{{}^{50}C_{17}(2)^{33}(a)^{17}} = \frac{a^{17}}{a^{16}}$

so, $a = \frac{{}^{50}C_{16} \times 2}{{}^{50}C_{17}} = \frac{50!}{16!34!} \times \frac{17!\,33!}{50!} \times 2 = 1$

Example: Find the coefficient of x^6y^3 in the expansion of $(x+2y)^9$.

Solution: Suppose x^6y^3 occurs in the $(r+1)^{th}$ term of the expansion $(x+2y)^9$

Now $T_{r+1} = {}^9C_r x^{9-r}(2y)^r = {}^9C_r 2^r x^{9-r} y^r$

Comparing the indices of x as well as in x^6y^3 and in T_{r+1}, we get $r=3$

Thus the coefficient of x^6y^3 is

${}^9C_3 2^3 = \frac{9!}{3!6!}.2^3 = \frac{9.8.7}{3.2}.2^3 = 672$